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Krzysztof Oleszkiewicz

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Jackson's theorem [**GR**, p. 35, eq. (2.6.2)], and (b) is a special case of Watson's q-analog of Whipple's theorem ([**GR**, p. 35, eq. (2.5.1)], see also [**A2**, p. 118, eq. (4.3)].) The discovery of (b) was motivated by [**S1**] and [**S2**].

We see fairly clearly how to do the 2-square theorem (a different instance of Jackson's theorem replaces (a)); however the theorems for 6 and 8 squares apparently require (using this approach) some instance of the ${}_6\Psi_6$ summation theorem [**GR**, p. 128, (5.3.1)] (see [**A1**, pp. 461–465] for details). Since we do not know a finitary analog of the ${}_6\Psi_6$ summation, the question of a similar result for 6 and 8 squares is of interest.

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Andrews:
Department of Mathematics
Pennsylvania State University
University Park, PA 16182
andrews@cantor.math.psu.edu

Ekhad & Zeilberger:
Department of Mathematics
Temple University
Philadelphia, PA 19122
ekhad@euclid.math.temple.edu
zeilberg@euclid.math.temple.edu

An Elementary Proof of Hilbert's Inequality

Krzysztof Oleszkiewicz

SUMMARY. We present an elementary proof of Hilbert's inequality (1) and we give a simple example showing that the constant in (1) is optimal. Moreover, we give a (slightly less elementary) extension of (1). There are a lot of applications of Hilbert's inequality to the theory of analytic functions and to the theory of functions of a real variable; some of them can be found in Chapter IX and Appendix 3 of Hardy, Littlewood and Polya's *Inequalities*, Cambridge 1964. The original proof of the inequality, due to Hilbert, is also given in that book.

Proposition 1 (Hilbert's inequality). *If* (a_m) *and* (b_n) *are square summable sequences of real numbers, then the double series* $\sum_{m,n=1}^{\infty} a_m b_n/(m+n)$ *is convergent* *and*

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \le \pi \sqrt{\sum_{m=1}^{\infty} a_m^2} \cdot \sqrt{\sum_{n=1}^{\infty} b_n^2}. \tag{1}$$

The inequality is strict unless one of the sequences (a_m) *or* (b_n) *is identically zero. Moreover,* π *is the best constant in* (1).

Proof of Proposition 1. We will use two lemmas.

Lemma 1. *For each positive number* m, $\sum_{n=1}^{\infty}(\sqrt{m}/\sqrt{n}(m+n)) < \pi$.

Proof: Let us denote points $(0,0), (0,\sqrt{m}), (\sqrt{m},\sqrt{n})$ by C, Y, X_n $(n = 0,1,2,\dots)$ respectively. Let S be the area of the quadrant of the circle centred at C with radius $\sqrt{m}$ from X_0 to Y. Let us denote by R_n the intersection of the circle and the line CX_n. Then let B_n be the intersection of the line CX_{n-1} and the vertical line containing the point R_n (for $n = 1,2,3,\dots$). Moreover, let S_n denote the area of the sector $R_{n-1}CR_n$ of the circle. (See picture 1.)

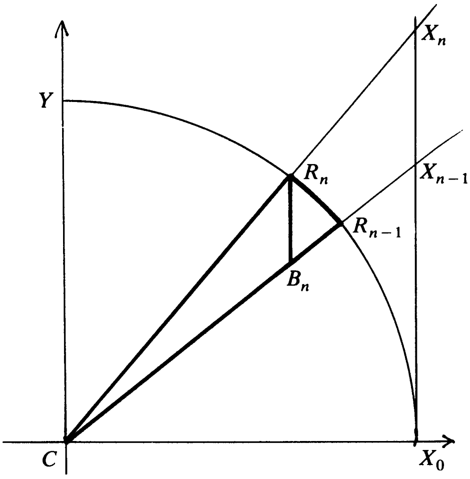

Figure 1

Denoting the area of the triangle KLM by $S_{\Delta KLM}$ we find

$$\begin{aligned}
\frac{\pi m}{4} = S = \sum_{n=1}^{\infty} S_n &> \sum_{n=1}^{\infty} S_{\Delta R_n C B_n} \\
&= \sum_{n=1}^{\infty} \left(\frac{|CR_n|}{|CX_n|}\right)^2 S_{\Delta X_{n-1} C X_n} \\
&= \sum_{n=1}^{\infty} \frac{m}{|CX_0|^2 + |X_0X_n|^2} \cdot \frac{|CX_0| \cdot |X_{n-1}X_n|}{2} \\
&= \sum_{n=1}^{\infty} \frac{m\sqrt{m}\left(\sqrt{n} - \sqrt{n-1}\right)}{2(m+n)} \\
&> \sum_{n=1}^{\infty} \frac{m\sqrt{m}}{4\sqrt{n}(m+n)}
\end{aligned}$$

and therefore

$$\sum_{n=1}^{\infty} \frac{\sqrt{m}}{\sqrt{n}\,(m+n)} < \pi. \quad \square$$

The generalization of Lemma 1 will be given later, in Lemma 3. Now we will prove Hilbert's inequality. Writing

$$\frac{a_m b_n}{m+n} = \frac{\sqrt[4]{m}}{\sqrt[4]{n}\,\sqrt{m+n}}\,a_m \cdot \frac{\sqrt[4]{n}}{\sqrt[4]{m}\,\sqrt{m+n}}\,b_n$$

and using Schwarz's inequality we find

$$\begin{aligned}
\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} &\leq \sqrt{\sum_{m,n=1}^{\infty} \frac{\sqrt{m}}{\sqrt{n}\,(m+n)}\,a_m^2} \cdot \sqrt{\sum_{m,n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{m}\,(m+n)}\,b_n^2} \\
&= \sqrt{\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{\sqrt{m}}{\sqrt{n}\,(m+n)} \right) a_m^2} \cdot \sqrt{\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{\sqrt{n}}{\sqrt{m}\,(m+n)} \right) b_n^2} \\
&\leq \pi \sqrt{\sum_{m=1}^{\infty} a_m^2} \cdot \sqrt{\sum_{n=1}^{\infty} b_n^2},
\end{aligned}$$

where Lemma 1 was used. Obviously the last inequality is strict unless one of the sequences (a_m) or (b_n) is identically zero. Now we will prove that π cannot be replaced by any smaller constant.

Lemma 2. *For each natural number $m > 1$*

$$\sum_{n=1}^{m-1} \frac{1}{\sqrt{mn}\,(m+n)} > \frac{\pi}{2m} - \frac{2}{m\sqrt{m}}.$$

Proof: Let us denote by A_n the intersection of the line CX_{n+1} and the line R_nB_n (for $n = 0, 1, \ldots, m-1$). Let S' be the area of the sector X_0CX_m of the circle. (See picture 2.) Then clearly

$$\begin{aligned}
\frac{\pi m}{8} = S' < \sum_{n=0}^{m-1} S_{\Delta R_n C A_n} &= \frac{\sqrt{m}}{2} + \sum_{n=1}^{m-1} \left(\frac{|CR_n|}{|CX_n|} \right)^2 S_{\Delta X_n C X_{n+1}} \\
&= \frac{\sqrt{m}}{2} + \sum_{n=1}^{m-1} \frac{m\sqrt{m}\,|X_nX_{n+1}|}{2\big(|CX_0|^2 + |X_0X_n|^2\big)} \\
&= \frac{\sqrt{m}}{2} + \sum_{n=1}^{m-1} \frac{m\sqrt{m}\,(\sqrt{n+1} - \sqrt{n})}{2(m+n)} \\
&< \frac{\sqrt{m}}{2} + \sum_{n=1}^{m-1} \frac{m\sqrt{m}}{4\sqrt{n}\,(m+n)}.
\end{aligned}$$

Thus

$$\sum_{n=1}^{m-1} \frac{1}{\sqrt{mn}\,(m+n)} > \frac{\pi}{2m} - \frac{2}{m\sqrt{m}}. \quad \square$$

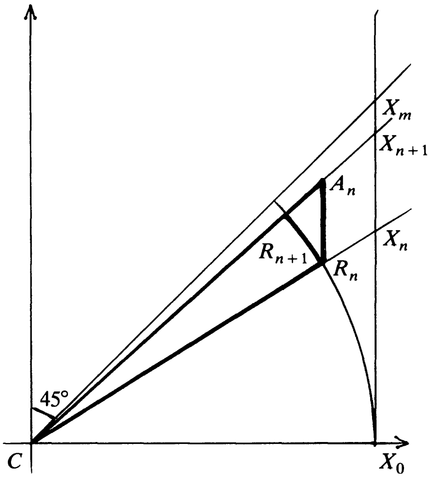

Figure 2

To prove that π is the best constant in Hilbert's inequality we will consider sequences $a_l = b_l = 1/\sqrt{l}$ for $l \leq k$, $a_l = b_l = 0$ or $l > k$, where k is a natural number. Then

$$\begin{aligned}
&\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \\
&\quad \geq \sum_{m=2}^{k} \left(\sum_{n=1}^{m-1} \frac{1}{\sqrt{mn}\,(m+n)} \right) + \sum_{n=2}^{k} \left(\sum_{m=1}^{n-1} \frac{1}{\sqrt{mn}\,(m+n)} \right) + \sum_{l=1}^{k} \frac{1}{2l^2} \\
&\quad \geq 2 \sum_{m=2}^{k} \left(\frac{\pi}{2m} - \frac{2}{m\sqrt{m}} \right) \qquad \text{(by Lemma 2)} \\
&\quad \geq \pi \sum_{m=1}^{k} \frac{1}{m} - \left(\pi + 4 \sum_{m=1}^{\infty} \frac{1}{m\sqrt{m}} \right)
\end{aligned}$$

and therefore

$$\frac{\displaystyle\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n}}{\sqrt{\displaystyle\sum_{m=1}^{\infty} a_m^2} \cdot \sqrt{\displaystyle\sum_{n=1}^{\infty} b_n^2}} \geq \pi - \frac{\pi + 4 \displaystyle\sum_{m=1}^{\infty} \frac{1}{m\sqrt{m}}}{\displaystyle\sum_{m=1}^{k} \frac{1}{m}} \xrightarrow{k \to \infty} \pi .$$

Hence π is the optimal constant in Hilbert's inequality. The proof of Proposition 1 is now complete. □

Now we will generalize Lemma 1.

Lemma 3. *For each positive number m and real number p > 1*

$$\sum_{n=1}^{\infty} \frac{m^{1/p}}{n^{1/p}(m+n)} \leq \frac{\pi}{\sin \dfrac{\pi}{p}} .$$

Proof: We can estimate

$$\sum_{n=1}^{\infty} \frac{m^{1/p}}{n^{1/p}(m+n)} \leq \int_0^{\infty} \frac{m^{1/p}\,dx}{x^{1/p}(x+m)} = \int_0^{\infty} \frac{dt}{t^{1/p}(1+t)}$$

$$= \int_0^1 \frac{dt}{t^{1/p}(1+t)} + \int_1^{\infty} \frac{dt}{t^{1+1/p}\left(1+\frac{1}{t}\right)}$$

$$= \int_0^1 \sum_{n=0}^{\infty} (-1)^n t^{n-1/p}\,dt + \int_1^{\infty} \sum_{n=0}^{\infty} (-1)^n t^{-n-1-1/p}\,dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^1 t^{n-1/p}\,dt + \sum_{n=0}^{\infty} (-1)^n \int_1^{\infty} t^{-n-1-1/p}\,dt$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{\frac{1}{p}-n} + p + \sum_{n=1}^{\infty} \frac{(-1)^n}{\frac{1}{p}+n} = \frac{\pi}{\sin\frac{\pi}{p}}.$$

To prove the last equality one can show that

$$\varphi(z) = \frac{\pi}{\sin \pi z} - \frac{1}{z} - \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{z-n} + \frac{1}{z+n}\right)$$

is an entire and bounded holomorphic function (we put $\varphi(z) = 0$ when z is an integer) and therefore (by Liouville's theorem) it is identically zero. For $z = 1/p$ we obtain the desired equality. There is also another way to prove Lemma 3—the integral $\int_0^{\infty} dt/t^{1/p}(1+t)$ can be done by contour integration (see chapter 3.6, the fourth type of integrals, H. Cartan *Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes*, Hermann, Paris, 1961). □

Therefore, using similar reasoning we can extend Proposition 1 as follows.

Proposition 2. *For* $p, q > 1$ *such that* $1/p + 1/q = 1$ *and sequences of non-negative numbers* $(a_m), (b_n)$ *such that* $\sum_{m=1}^{\infty} a_m^p, \sum_{n=1}^{\infty} b_n^q$ *are convergent*

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{m=1}^{\infty} a_m^p\right)^{1/p} \left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}. \tag{2}$$

The proof is essentially as above (we use Hölder's inequality instead of Schwarz's inequality). The constant in (2) is also optimal.

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Instytut Matematyki
Uniwersytet Warszawski
ul. Banacha 2
02-097 Warszawa, Poland