An Elementary Proof of # # n = 1 1/n2 = #2/6

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Following §2.2 the eigenvectors corresponding to λ are given by

$$\mathbf{x}_j = \Big[k_j, 0, 0, \ldots, \underset{j\text{th position}}{-k_1}, 0, 0, \ldots, 0\Big]^T, \qquad j = 2, 3, \ldots, n$$

and corresponding to μ the eigenvector is

$$\mathbf{x}_1 = \mathbf{k} = [k_1, k_2, \ldots, k_n]^T.$$

As an example with $\mathbf{k} = [1, -1, 2]^T$, $a = 2$ and $\lambda = -7$ we get

$$A = 2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}[1, -1, 2] - 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -2 & 4 \\ -2 & -5 & -4 \\ 4 & -4 & 1 \end{bmatrix}$$

a matrix with eigenvalues $5, -7, -7$ and corresponding eigenvectors $[1, -1, 2]^T$, $[1, 1, 0]^T$ and $[2, 0, -1]^T$.

Some general results relating to symmetric matrices with specified eigenvalues and eigenvectors may be found in [**6**].

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An Elementary Proof of $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$

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Euler's formula given in the title is one of those formulas in mathematics that can be proved in many different ways. For example, one can prove it by using Cauchy's Residue Calculus ([**1**], [**2**], [**3**], [**4**]) or Weierstrass' Product Theorem ([**5**]) in the theory of functions of one complex variable. Also, many advanced calculus books present it as an exercise so that one can prove it by means of Parseval's Theorem ([**6**], [**7**]) or Fourier series expansions ([**6**], [**7**], [**8**], [**9**]). In addition, there are many papers that contain elementary proofs; several are referenced in the article [**10**].

The purpose of this note is to give another elementary proof of this formula. Our proof is more elementary, as the following calculations show.

First of all, since $\sum_{n=1}^{\infty} 1/n^2 = \sum_{n=0}^{\infty} 1/(2n+1)^2 + \sum_{n=1}^{\infty} 1/(2n)^2$ (by absolute convergence), we only need verify that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}. \tag{1}$$

Let us consider the Taylor series expansion of arcsin x near $x = 0$, which we can easily derive from the binomial series expansion of $(1 - x^2)^{-1/2}$ near $x = 0$,

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{x^{2n+1}}{2n+1}. \tag{2}$$

The series in (2) converges at $x = 1$ by Raabe's test, hence converges uniformly on $[-1, 1]$ by Weierstrass' M-test. Accordingly, the series in (2) actually represents arcsin x on $[-1, 1]$ by Abel's Theorem.

Substitute $x = \sin t$ into both sides of (2) to obtain

$$t = \sin t + \sum_{n=1}^{\infty} \frac{1}{2n+1} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \sin^{2n+1} t, \tag{3}$$

where $-\pi/2 \leqslant t \leqslant \pi/2$. Integrating both sides of (3), from 0 to $\pi/2$, term by term, we find

$$\frac{\pi^2}{8} = 1 + \sum_{n=1}^{\infty} \frac{1}{2n+1} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \int_0^{\pi/2} \sin^{2n+1} t \, dt. \tag{4}$$

Moreover, by Wallis' formula ([**11**], [**12**]) we have

$$\int_0^{\pi/2} \sin^{2n+1} t \, dt = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \quad \text{for } n = 1, 2, 3, \ldots. \tag{5}$$

Finally, (4), together with (5), shows (1).

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