



An Elementary Proof of $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Boo Rim Choe

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Following §2.2 the eigenvectors corresponding to λ are given by

$$\mathbf{x}_j = \left[k_j, 0, 0, \dots, \underset{j\text{th position}}{-k_1}, 0, 0, \dots, 0 \right]^T, \quad j = 2, 3, \dots, n$$

and corresponding to μ the eigenvector is

$$\mathbf{x}_1 = \mathbf{k} = [k_1, k_2, \dots, k_n]^T.$$

As an example with $\mathbf{k} = [1, -1, 2]^T$, $a = 2$ and $\lambda = -7$ we get

$$A = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} [1, -1, 2] - 7 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -2 & 4 \\ -2 & -5 & -4 \\ 4 & -4 & 1 \end{bmatrix}$$

a matrix with eigenvalues 5, -7, -7 and corresponding eigenvectors $[1, -1, 2]^T$, $[1, 1, 0]^T$ and $[2, 0, -1]^T$.

Some general results relating to symmetric matrices with specified eigenvalues and eigenvectors may be found in [6].

REFERENCES

1. J. C. Renaud, Matrices with Integer Elements and Integer Eigenvalues, *The American Mathematical Monthly*, 90 (March 1983).
2. L. Mirsky, *An Introduction to Linear Algebra*, O.U.P., 1955.
3. P. Lancaster, *Theory of Matrices*, Academic Press, 1969.
4. M. Newman, *Integral Matrices*, Academic Press, New York, 1972.
5. J. Wilkinson, *The Algebraic Eigenvalue Problem*, Ch. 9, §20, O.U.P., 1965.
6. K. J. Heuvers, Symmetric Matrices with Prescribed Eigenvalues and Eigenvectors, *Mathematics Magazine*, 55 (March 1982).

An Elementary Proof of $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$

BOO RIM CHOE

Department of Mathematics, University of Wisconsin, Madison, WI 53706

Euler's formula given in the title is one of those formulas in mathematics that can be proved in many different ways. For example, one can prove it by using Cauchy's Residue Calculus ([1], [2], [3], [4]) or Weierstrass' Product Theorem ([5]) in the theory of functions of one complex variable. Also, many advanced calculus books present it as an exercise so that one can prove it by means of Parseval's Theorem ([6], [7]) or Fourier series expansions ([6], [7], [8], [9]). In addition, there are many papers that contain elementary proofs; several are referenced in the article [10].

The purpose of this note is to give another elementary proof of this formula. Our proof is more elementary, as the following calculations show.

First of all, since $\sum_{n=1}^{\infty} 1/n^2 = \sum_{n=0}^{\infty} 1/(2n+1)^2 + \sum_{n=1}^{\infty} 1/(2n)^2$ (by absolute convergence), we only need verify that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}. \quad (1)$$

Let us consider the Taylor series expansion of $\arcsin x$ near $x = 0$, which we can easily derive from the binomial series expansion of $(1-x^2)^{-1/2}$ near $x = 0$,

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{x^{2n+1}}{2n+1}. \quad (2)$$

The series in (2) converges at $x = 1$ by Raabe's test, hence converges uniformly on $[-1, 1]$ by Weierstrass' M -test. Accordingly, the series in (2) actually represents $\arcsin x$ on $[-1, 1]$ by Abel's Theorem.

Substitute $x = \sin t$ into both sides of (2) to obtain

$$t = \sin t + \sum_{n=1}^{\infty} \frac{1}{2n+1} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \sin^{2n+1} t, \quad (3)$$

where $-\pi/2 \leq t \leq \pi/2$. Integrating both sides of (3), from 0 to $\pi/2$, term by term, we find

$$\frac{\pi^2}{8} = 1 + \sum_{n=1}^{\infty} \frac{1}{2n+1} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \int_0^{\pi/2} \sin^{2n+1} t \, dt. \quad (4)$$

Moreover, by Wallis' formula ([11], [12]) we have

$$\int_0^{\pi/2} \sin^{2n+1} t \, dt = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \quad \text{for } n = 1, 2, 3, \dots \quad (5)$$

Finally, (4), together with (5), shows (1).

REFERENCES

1. Jerrold E. Marsden, *Basic Complex Analysis*, W. H. Freeman, San Francisco, 1973, pp. 247–252.
2. J. H. Curtiss, *Introduction to The Functions of a Complex Variable*, Marcell Dekker, New York, 1978, p. 267.
3. Lars V. Ahlfors, *Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1979, p. 190.
4. John B. Conway, *Functions of One Complex Variable*, 2nd ed., Springer-Verlag, New York, 1978, p. 122.
5. Herb Silverman, *Complex Variables*, Houghton Mifflin, New York, 1975, p. 359.
6. Walter Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill, New York, 1976, pp. 198–199.
7. Robert G. Bartle, *The Elements of Real Analysis*, 2nd ed., Wiley, New York, 1976, pp. 341–343.
8. Watson Fulks, *Advanced Calculus*, 3rd ed., Wiley, New York, 1978, p. 663.
9. M. H. Protter and C. B. Morrey, *A First Course in Real Analysis*, Springer-Verlag, New York, 1977, p. 268.
10. E. L. Stark, The Series $\sum_{k=1}^{\infty} k^{-s}$, $s = 2, 3, 4, \dots$ Once More, *Mathematics Magazine*, 47 (1974) 197–202.
11. H. Anton, *Calculus with Analytic Geometry*, 2nd ed., Wiley, New York, 1984, p. 490.
12. John A. Tierney, *Calculus and Analytic Geometry*, 3rd ed., Allyn and Bacon, Boston, 1975, p. 378.