1 Introduction

We will now take a somewhat deeper look at the quantization of analog-valued, discrete-time sources. Surprisingly, we will find that the combination of a uniform scalar quantizer followed by entropy coding is nearly optimum within the class of scalar quantizers.

Throughout this lecture, we assume that the source output is a sequence $U_1, U_2, \ldots$, of iid real analog-valued random variables, each with a probability density $f_{U}(u)$. We also assume that the probability density function (pdf) $f_{U}(u)$ is smooth enough and the quantization fine enough that $f_{U}(u)$ is almost constant over each quantization region.

The analogue of the entropy $H[X]$ of a discrete random variable is the differential entropy $h[U]$ of an analog random variable. After defining $h[U]$, we will compare and contrast the properties of $H[U]$ and $h[U]$.

We will then analyze the performance of a uniform scalar quantizer followed by entropy coding. We will see that there is a tradeoff between the rate of the quantizer and the mean-squared error (MSE) between source and quantized output. We also show that the uniform quantizer is essentially optimum among scalar quantizers at high-rate.

We similarly analyze the performance of uniform vector quantizers followed by entropy coding, and find similar tradeoffs. We will find that vector quantizers can achieve a gain over scalar quantizers (i.e., a reduction of MSE for given quantizer rate), but that the reduction in MSE is at most a factor of $\pi e/6 = 1.42$.

As in the discrete case, generalizations to continuous sources with memory are possible, but these are not discussed.

2 Differential entropy

The differential entropy $h[U]$ of a continuous-valued random variable $U$ is analogous to the entropy $H[X]$ of a discrete chance variable $X$. It has many similarities, but also some important differences.

**Definition** The differential entropy of a continuous-valued real random variable $U$ with pdf $f_{U}(u)$ is

$$h[U] = \int_{-\infty}^{\infty} -f_{U}(u) \log f_{U}(u) \, du.$$ 

The integral may be restricted to the region where $f_{U}(u) > 0$ since $0 \log 0$ is interpreted to be equal to 0. We assume that $f_{U}(u)$ is smooth, well-behaved, and that the integral is finite.
As before, we take the base of the logarithm to be 2 and take the units of \( h[U] \) to be bits/symbol.

Like \( H[X] \), the differential entropy \( h[U] \) is the expected value of the random variable 
\[-\log f_U(U). \] The log of the joint density of several independent random variables is the sum of the logs of the individual pdf’s, and this can be used to derive an AEP similar to the discrete case. We will not pursue the AEP here however.

Unlike \( H[X] \), the differential entropy \( h[U] \) can be negative and depends on the scaling of the outcomes. This can be seen from the following two examples.

**Example 1** (*uniform distributions*). Let \( f_U(u) \) be a uniform distribution over an interval \([a, a + \Delta]\) of length \( \Delta \); i.e., \( f_U(u) = \frac{1}{\Delta} \) for \( u \in [a, a + \Delta] \), and \( f_U(u) = 0 \) elsewhere. Then 
\[-\log f_U(u) = \log \Delta \text{ where } f_U(u) > 0 \] and 
\[ h[U] = E[-\log f_U(U)] = \log \Delta. \]

**Example 2** (*Gaussian distributions*). Let \( f_U(u) \) be a Gaussian distribution with mean \( m \) and variance \( \sigma^2 \); i.e.,
\[ f_U(u) = \sqrt{\frac{1}{2\pi\sigma^2}} \exp \left\{ -\frac{(u - m)^2}{2\sigma^2} \right\}. \]
Then 
\[-\log f_U(u) = \frac{1}{2} \log 2\pi\sigma^2 + (\log e)(u - m)^2/(2\sigma^2). \] Since \( E[(U - m)^2] = \sigma^2 \), we have 
\[ h[U] = E[-\log f_U(U)] = \frac{1}{2} \log(2\pi\sigma^2) + \frac{1}{2} \log e = \frac{1}{2} \log(2\pi e\sigma^2). \]

It can be seen from these expressions that by making \( \Delta \) or \( \sigma^2 \) arbitrarily small, the differential entropy can be made arbitrarily negative, while by making \( \Delta \) or \( \sigma^2 \) arbitrarily large, the differential entropy can be made arbitrarily positive.

Moreover, if we rescale the random variable \( U \) to \( \alpha U \) for some scale factor \( \alpha > 0 \), then the differential entropy is increased by \( \log \alpha \), in these examples and in general. In other words, \( h[U] \) is not invariant to scaling.

One of the important properties of entropy is that it does not depend on the labeling of the elements of the alphabet, i.e., it is invariant to invertible transformations. The differential entropy is very different in this respect, and, as we have just seen, it is modified by even such a trivial transformation as a change of scale. The reason for this is that the probability density is a probability per unit length, and therefore depends on the measure of length. In fact, as we see more clearly later, this fits in very well with the fact that source coding for analog sources also depends on an error term per unit length.

Like the entropy, the differential entropy has the property that if \( U \) and \( V \) are independent random variables, then the entropy of the joint variable \( UV \) with pdf \( f_{UV}(u,v) = f_U(u)f_V(v) \) is \( h[UV] = h[U] + h[V] \). Again, this follows from the fact that the log of the probability density is additive, i.e., 
\[-\log f_{UV}(u,v) = -\log f_U(u) - \log f_V(v). \]
Thus the differential entropy of a vector random variable \( U^n \), corresponding to a string of \( n \) iid random variables \( U_1, U_2, \ldots, U_n \), each with the density \( f_U(u) \), is 
\[ h[U^n] = nh[U]. \]
3 Performance of high-rate scalar quantizers

We first analyze uniform scalar quantizers and then show that non-uniform quantizers have no advantages over the uniform case in the high-rate limit.

3.1 Performance of uniform high-rate scalar quantizers

In a uniform scalar quantizer, every quantization interval $\mathcal{R}_j$ has the same length $|\mathcal{R}_j| = \Delta$. In other words, $\mathbb{R}$ (or the portion of $\mathbb{R}$ over which $f_U(u) > 0$), is partitioned into equal length intervals, each of length $\Delta$.

$$\mathcal{R}_{-1} \quad \mathcal{R}_0 \quad \mathcal{R}_1 \quad \mathcal{R}_2 \quad \mathcal{R}_3 \quad \mathcal{R}_4 \quad \cdots$$

Figure 1: Uniform scalar quantizer

We assume that there are enough quantization levels to cover the region where $f_U(u) > 0$. For example 2 above, this requires an infinite number of levels, $-\infty < j < \infty$. This is an example where the quantized discrete random variable $V$ has a countably infinite alphabet. Obviously, practical quantizers limit the number of levels to a finite region $\mathcal{R}$ such that $\int_{\mathcal{R}} f_U(u) \, du \approx 1$.

The high-rate assumption is that $\Delta$ is small enough so that the pdf $f_U(u)$ is approximately constant over any one quantization interval. More precisely, define $\bar{f}(u)$ (see Figure 2) as the average value of $f_U(u)$ over the quantization interval containing $u$,

$$\bar{f}(u) = \frac{\int_{\mathcal{R}_j} f_U(u) \, du}{\Delta} \quad \text{for} \quad u \in \mathcal{R}_j \quad (1)$$

We see from (1) that for each $j$, $\Delta \bar{f}(u) = \Pr(\mathcal{R}_j)$ for $u \in \mathcal{R}_j$.

$$\bar{f}(u) \quad f_U(u)$$

Figure 2: Average density over each $\mathcal{R}_j$

The high-rate assumption is that $f_U(u) \approx \bar{f}(u)$ for all $u \in \mathbb{R}$. This means that $f_U(u) \approx \Pr(\mathcal{R}_j)/\Delta$ for $u \in \mathcal{R}_j$. It also means that the conditional pdf $f_{U|\mathcal{R}_j}(u)$ of $U$ conditional on $u \in \mathcal{R}_j$ is approximated by

$$f_{U|\mathcal{R}_j}(u) \approx \begin{cases} 1/\Delta, & u \in \mathcal{R}_j; \\ 0, & u \notin \mathcal{R}_j. \end{cases}$$
Consequently the conditional mean $a_j$ is approximately in the center of the interval $\mathcal{R}_j$, and the mean-squared error is approximately given by

$$\text{MSE} \approx \int_{-\Delta/2}^{\Delta/2} \frac{1}{\Delta} u^2 du = \frac{\Delta^2}{12}, \quad (2)$$

for each quantization interval $\mathcal{R}_j$. Consequently this is also the overall MSE.

We next analyze the entropy of the quantizer output $V$. The probability $p_j$ that $V = a_j$ is given by both

$$p_j = \int_{\mathcal{R}_j} f_U(u) \, du \quad \text{and, for all } u \in \mathcal{R}_j, \quad p_j = \bar{f}(u)\Delta. \quad (3)$$

Therefore the entropy of the discrete random variable $V$ is

$$H[V] = \sum_j -p_j \log p_j = \sum_j \int_{\mathcal{R}_j} -f_U(u) \log[\bar{f}(u)\Delta] \, du$$

$$= \int_{-\infty}^{\infty} -f_U(u) \log[\bar{f}(u)\Delta] \, du \quad (4)$$

$$= \int_{-\infty}^{\infty} -f_U(u) \log[\bar{f}(u)] \, du - \Delta$$

where we have first combined the sum of disjoint integrals into a single integral and then used $\log[\bar{f}(u)\Delta] = \log \bar{f}(u) + \log \Delta$.

Finally, using the high-rate approximation$^1$ $f_U(u) \approx \bar{f}(u)$, this becomes

$$H[V] \approx \int_{-\infty}^{\infty} -f_U(u) \log[f_U(u)\Delta] \, du$$

$$= h[U] - \log \Delta \quad (6)$$

Since the sequence $U_1, U_2, \ldots$ of inputs to the quantizer is memoryless (iid), the quantizer output sequence $V_1, V_2, \ldots$ is an iid sequence of discrete chance variables representing quantization levels—i.e., a discrete memoryless source. A uniquely decodable source code can therefore be used to encode this output sequence into a bit sequence at an average rate of $\bar{L} \approx H[V] \approx h[U] - \log \Delta$ bits/symbol. At the receiver, the mean-squared quantization error in reconstructing the original sequence will be approximately $\text{MSE} \approx \Delta^2/12$.

The important conclusions from this analysis are illustrated in Figure 3 and are summarized as follows:

- Under the high-rate assumption, the rate $\bar{L}$ for a uniform quantizer followed by discrete entropy coding depends only on the differential entropy $h[U]$ of the source.

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$^1$Problem 4.5 provides some insight into the nature of the approximation here. In particular, the difference between $h[U] - \log \Delta$ and $H[V]$ is $\int f_U(u) \log[\bar{f}(u)/f_U(u)] \, du$. This quantity is always non-positive and goes to zero with $\Delta$ as $\Delta^2$. Similarly, the approximation error on the MSE goes to 0 as $\Delta^4$. 

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and the spacing $\Delta$ of the quantizer. It does not depend on any other feature of the source pdf $f_U(u)$, nor on any other feature of the quantizer, such as the number $M$ of levels, so long as the quantizer intervals cover $f_U(u)$ sufficiently completely and finely.

- The rate $\bar{L} \approx H[V]$ and the MSE are parametrically related by $\Delta$, i.e.,

$$\bar{L} \approx h(U) - \log \Delta; \quad \text{MSE} \approx \frac{\Delta^2}{12} \quad (7)$$

Note that each reduction in $\Delta$ by a factor of 2 will reduce the MSE by a factor of 4 and increase the required transmission rate $\bar{L} \approx H[V]$ by 1 bit/symbol. Communication engineers express this by saying that each additional bit per symbol decreases the mean-squared distortion$^2$ by 6 dB. Figure 3 sketches MSE as a function of $\bar{L}$.

![Figure 3: MSE as a function of $\bar{L}$ for a scalar quantizer with the high-rate approximation.](image)

Note that changing the source entropy $h(U)$ simply shifts the figure right or left. Note also that log MSE is linear, with a slope of -2, as a function of $\bar{L}$.

Conventional $b$-bit analog-to-digital (A/D) converters are uniform scalar $2^b$-level quantizers that cover a certain range $\mathcal{R}$ with a quantizer spacing $\Delta = 2^{-b}|\mathcal{R}|$. In practice, A/D converters do not use entropy coding. The input samples must be scaled so that the probability that $u / \in \mathcal{R}$ (the “overflow probability”) is small. For a fixed scaling of the input, the tradeoff is again that increasing $b$ by 1 bit reduces the MSE by a factor of 4.

Finally, note that the relation $H[V] \approx h[u] - \log \Delta$ helps to explain the nature of the differential entropy $h(U)$. $H[V]$ is the entropy of a finely quantized version of $U$, and the additional term $\log \Delta$ relates to the “uncertainty” within an individual quantized interval. It shows explicitly how the scale used to measure $U$ affects $h[U]$.

## 3.2 The performance of non-uniform scalar quantizers

In this section, we show that the analysis of uniform scalar quantizers in the last section also provides an approximate lower bound on the MSE for any non-uniform scalar quantizer, again using the high-rate approximation that the pdf of $U$ is constant within each quantization region. This shows that in the high-rate region, there is little reason to further consider non-uniform scalar quantizers.

$^2$A quantity $x$ expressed in dB is defined to be $10 \log_{10} x$. We discuss this very useful and common logarithmic measure later.
Consider an arbitrary scalar quantizer for a random variable $U$ with a pdf $f_U(u)$. Let $\Delta_j$ be the width of the $j$th quantization interval, i.e., $\Delta_j = |R_j|$. As before, let $\overline{f}(u)$ be the average pdf within each quantization interval, i.e.,

$$\overline{f}(u) = \frac{\int_{R_j} f_U(u) \, du}{\Delta_j} \quad \text{for} \quad u \in R_j$$

The high-rate approximation is that $f_U(u)$ is approximately constant over each quantization region. Equivalently, $f_U(u) \approx \overline{f}(u)$ for all $u$. Thus, if region $R_j$ has width $\Delta_j$, the conditional mean $a_j$ of $U$ over $R_j$ is approximately the midpoint of the region, and the conditional mean-squared error, MSE$_j$, given $U \in R_j$, is approximately $\Delta_j^2 / 12$.

Let $V$ be the quantizer output, i.e., the discrete random variable such that $V = a_j$ for $U \in R_j$. The probability $p_j$ that $V = a_j$ is

$$p_j = \frac{\int_{R_j} f_U(u) \, du}{\Delta_j}$$

The unconditional mean-squared error, i.e., $E[(U - V)^2]$ is then given by

$$\text{MSE} \approx \sum_j p_j \frac{\Delta_j^2}{12} = \sum_j \int_{R_j} f_U(u) \frac{\Delta_j^2}{12} \, du.$$  \hfill (8)

We can simplify this by defining $\Delta(u) = \Delta_j$ for $u \in R_j$. Since each $u$ is in $R_j$ for some $j$, this defines $\Delta(u)$ for all $u \in \mathbb{R}$. Substituting this in (8),

$$\text{MSE} \approx \sum_j \int_{R_j} f_U(u) \frac{\Delta(u)^2}{12} \, du$$

$$= \int_{-\infty}^{\infty} f_U(u) \frac{\Delta(u)^2}{12} \, du.$$  \hfill (10)

We next analyze the entropy of $V$. As in (3), we use the following relations for $p_j$

$$p_j = \int_{R_j} f_U(u) \, du \quad \text{and, for all} \quad u \in R_j, \quad p_j = \overline{f}(u)\Delta(u)$$

$$H[V] = \sum_j -p_j \log p_j$$

$$= \sum_j \int_{R_j} -f_U(u) \log[\overline{f}(u)\Delta(u)] \, du$$

$$= \int_{-\infty}^{\infty} -f_U(u) \log[\overline{f}(u)\Delta(u)] \, du$$

where we have combined the multiple integrals over disjoint regions into a single integral. We now substitute the high-rate approximation $f_U(u) \approx \overline{f}(u)$ into (12),

$$H[V] \approx \int_{-\infty}^{\infty} -f_U(u) \log[f_U(u)\Delta(u)] \, du$$

$$= h[U] - \int_{-\infty}^{\infty} f_U(u) \log \Delta(u) \, du.$$  \hfill (13)
Note the similarity of this to (6).

We are now faced with the problem of minimizing the mean-squared error subject to a constraint on the entropy $H[V]$. Instead we minimize the approximation to MSE in (13) subject to the approximation to $H[V]$ in (10). Problem 4.5 provides some insight into the accuracy of these approximations.

We use the Lagrange multiplier method to perform this minimization. Since MSE decreases as $H[V]$ increases, we consider minimizing $\text{MSE} + \lambda H[V]$ where as $\lambda$ increases, we expect MSE to increase and $H[V]$ to decrease in the minimizing solution.

In principle, the minimization should be constrained by the fact that $\Delta(u)$ is constrained to represent the interval sizes for a realizable set of quantization regions. We will lower bound the minimum of $\text{MSE} + \lambda H[V]$ by ignoring this constraint. The very nice thing that happens is that this unconstrained lower bound occurs where $\Delta(u)$ is constant. This corresponds to a uniform quantizer, which is clearly realizable. In other words, subject to the high-rate approximation, the lower bound on MSE over all scalar quantizers is equal to the MSE for the uniform scalar quantizer. To see this, we use (10) and (13),

$$\text{MSE} + \lambda H[V] \approx \int_{-\infty}^{\infty} f_U(u) \frac{\Delta(u)^2}{12} \, du + \lambda h[U] - \lambda \int_{-\infty}^{\infty} f_U(u) \log \Delta(u) \, du$$

$$= \lambda h[U] + \int_{-\infty}^{\infty} f_U(u) \left\{ \frac{\Delta(u)^2}{12} - \lambda \log \Delta(u) \right\} \, du$$

(14)

We can minimize this over all choices of $\Delta(u) > 0$ by simply minimizing the expression inside the braces for each real value of $u$. That is, for each $u$, we can differentiate the quantity inside the braces with respect to $\Delta(u)$, getting $\Delta(u)/6 - \lambda (\log e)/\Delta(u)$. Setting the derivative equal to 0, we see that $\Delta(u) = \sqrt{\lambda (\log e)/6}$. By taking the second derivative, we see that this solution actually minimizes the integrand for each $u$. The only thing important here is that the minimizing $\Delta(u)$ is independent of $u$. This means that the approximation of MSE is minimized, subject to a constraint on the approximation of $H[V]$, by the use of a uniform quantizer.

We must now question what it means to minimize an approximation to something subject to a constraint which itself is an approximation. From Problem 4.5, we see that both the approximation to MSE and that to $H[V]$ are good approximations for small $\Delta$, i.e., for high-rate. For any given high-rate non-uniform quantizer then, consider plotting MSE and $H[V]$ on Figure 3. The corresponding approximate values of MSE and $H[V]$ are then close to the plotted value (with some small difference both in the ordinate and abscissa). These approximate values, however, lie above the approximate values plotted in Figure 3 for the scalar quantizer. Thus, in this sense, the performance curve of MSE versus $H[V]$ for the approximation to the scalar quantizer either lies below or close to any point for any non-uniform quantizer.

In summary, we have shown that for large $H[V]$ (i.e., high-rate quantization), a uniform scalar quantizer approximately minimizes MSE subject to the entropy constraint. There is little reason to use non-uniform scalar quantizers (except perhaps at low rate). Furthermore the MSE performance at high-rate can be easily approximated and depends only on $h[U]$ and the constraint on $H[V]$. 

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4 High-rate two-dimensional quantizers

We now give a similar approximate analysis of the performance of a high-rate two-dimensional (2D) quantizer. As in the scalar case, we start with uniform 2D quantizers and then argue that, in the high-rate region, these minimize the MSE for any given constraint on the quantizer output entropy.

A 2D quantizer operates on 2 source samples \( u = (u_1, u_2) \) at a time; i.e., the source alphabet is \( U^2 = \mathbb{R}^2 \). Assuming iid source symbols, the joint pdf is then \( f_U(u) = f_U(u_1)f_U(u_2) \), and the joint differential entropy is \( h[U^2] = 2h[U] \).

4.1 Uniform 2D quantizers

Like a uniform scalar quantizer, a uniform 2D quantizer is based on a fundamental quantization region \( Q \) (“quantization cell”) whose translates tile\(^3\) the 2D plane. In the one-dimensional case, there is really only one sensible choice for \( Q \), namely an interval of length \( \Delta \), but in higher dimensions there are many possible choices. For two dimensions, the most important choices are squares and hexagons, but in higher dimensions, many more choices are available.

Notice that if a region \( Q \) tiles \( \mathbb{R}^2 \), then any scaled version \( \alpha Q \) of \( Q \) will tile \( \mathbb{R}^2 \), and so will any rotation or translation of \( Q \).

We may now analyze the performance of a uniform 2D quantizer with basic cell \( Q \), which we assume is centered at the origin \( 0 \). The set of cells, which are assumed to tile the region, are denoted by \( \{Q_j\} \) where \( Q_j = a_j + Q \) and \( a_j \) is the center of the cell \( Q_j \). Let \( A(Q) = \int_Q du \) be the area of the basic cell. The average pdf in a cell \( Q_j \) containing \( u \). We again make the high-rate assumption that the region \( Q \) is small enough that \( f_U(u) \approx \bar{f}(u) \) for all \( u \).

The assumption \( f_U(u) \approx \bar{f}(u) \) implies that the conditional pdf, conditional on \( u \in Q_j \) is approximated by

\[
f_{U|Q_j}(u) \approx \begin{cases} \frac{1}{A(Q)}, & u \in Q_j; \\ 0, & u \notin Q_j. \end{cases}
\]  

(15)

The conditional mean is approximately equal to the center \( a_j \) of the region \( Q_j \). The mean-squared error per dimension for the basic quantization cell \( Q \) centered on 0 is then approximately equal to

\[
MSE \approx \frac{1}{2} \int_Q \|u\|^2 \frac{1}{A(Q)} du.
\]  

(16)

\(^3\)A region of the 2D plane is said to tile the plane if the region, plus translates and rotations of the region, fill the plane without overlap. For example the square and the hexagon tile the plane. Also rectangles tile the plane and equilateral triangles with rotations tile the plane.
The right side of (16) is the MSE for the quantization area $Q$ using a pdf equal to a constant; it will be denoted $\text{MSE}_c$. The quantity $\|u\|$ is the length of the vector $u_1, u_2$, so that $\|u\|^2 = u_1^2 + u_2^2$. Thus $\text{MSE}_c$ can be rewritten as

$$\text{MSE} \approx \text{MSE}_c = \frac{1}{2} \int_Q (u_1^2 + u_2^2) \frac{1}{A(Q)} \, du_1 du_2. \quad (17)$$

$\text{MSE}_c$ is measured in units of squared length, just like $A(Q)$. Thus the ratio $G(Q) = \text{MSE}_c / A(Q)$ is a dimensionless quantity called the normalized second moment. With a little effort, it can be seen that $G(Q)$ is invariant to scaling, translation and rotation. $G(Q)$ does depend on the shape of the region $Q$, and, as we see below, it is $G(Q)$ that determines how well a given shape performs as a quantization region. By expressing

$$\text{MSE}_c = G(Q) A(Q),$$

we see that the MSE is the product of a shape term and an area term, and these can be chosen independently.

As examples, we compute $G(Q)$ for some common shapes.

- **Square**: For a square $\Delta$ on a side, $A(Q) = \Delta^2$. Breaking (17) into two terms, we see that each is identical to the scalar case and $\text{MSE}_c = \Delta^2 / 12$. Thus $G(\text{Square}) = 1/12$.

- **Hexagon**: View the hexagon as the union of 6 equilateral triangles $\Delta$ on a side. Then $A(Q) = 3\sqrt{3}\Delta^2 / 2$ and $\text{MSE}_c = 5\Delta^2 / 24$. Thus $G(\text{hexagon}) = 5/(36\sqrt{3})$.

- **Circle**: For a circle of radius $r$, $A(Q) = \pi r^2$ and $\text{MSE}_c = r^2 / 4$ so $G(\text{circle}) = 1/(4\pi)$.

The circle is not an allowable quantization region since it does not tile the plane. On the other hand, for a given area, this is the shape that minimizes $\text{MSE}_c$. To see this, note that for any other shape, differential areas further from the origin can be moved closer to the origin with a reduction in $\text{MSE}_c$. That is, the circle is the 2D shape that minimizes $G(Q)$. We also see from this why $G(\text{Hexagon}) < G(\text{Square})$, since the hexagon is more concentrated around the origin than the square.

Returning to (15) and (16), we see that each quantization cell $Q_j$ has the same shape and has a conditional pdf which is approximately uniform. Thus $\text{MSE}_c$ approximates the MSE for each of the quantization regions for a uniform quantizer under the high-rate assumption.

We now turn to computing the entropy of the quantizer output. The probability that $U$ falls in the region $Q_j$ is

$$p_j = \int_{Q_j} f_U(u) \, du \quad \text{and, for all } u \in Q_j, \quad p_j = \frac{f(u)}{A(Q)}$$

The output of the quantizer is the discrete chance variable $V$ with the pmf $\{p_j\}$. As
before, the entropy of $V$ is given by
\[
H[V] = -\sum_j p_j \log p_j
\]

\[
= -\sum_j \int_{Q_j} f_U(u) \log[\overline{f}(u)A(Q)] \, du
\]

\[
= -\int f_U(u) \left[ \log \overline{f}(u) + \log A(Q) \right] \, du
\]

\[
\approx -\int f_U(u) \left[ \log f_U(u) \right] \, du + \log A(Q)
\]

\[
= 2h[U] - \log A(Q),
\]

where we use the assumptions that $f_U(u)$ is smooth and $A(Q)$ is small to justify the approximation $f_U(u) \approx \overline{f}(u)$. Note that, since $U = U_1 U_2$ for iid variables $U_1$ and $U_2$, the differential entropy of $U$ is $2h[U]$.

Again, an efficient uniquely decodable source code can be used to encode the quantizer output sequence into a bit sequence at an average rate per source symbol of

\[
\bar{L} \approx \frac{H[V]}{2} \approx h[U] - \frac{1}{2} \log A(Q) \quad \text{bits/symbol.} \tag{18}
\]

At the receiver, the mean-squared quantization error in reconstructing the original sequence will be approximately equal to the MSE given in (16).

We have the following important conclusions for a uniform 2D quantizer under the high-rate approximation:

- Under the high-rate assumption, the rate $\bar{L}$ depends only on the differential entropy $h[U]$ of the source and the area $A(Q)$ of the basic quantization cell $Q$. It does not depend on any other feature of the source pdf $f_U(u)$, and does not depend on the shape of the quantizer region, i.e., it does not depend on the normalized second moment $G(Q)$.

- There is a tradeoff between the rate $\bar{L}$ and MSE that is governed by the area $A(Q)$. From (18), an increase of 1 bit/symbol in rate corresponds to a decrease in $A(Q)$ by a factor of 4. From (16), this decreases the MSE by a factor of 4, i.e., by 6 dB.

- The ratio $G(\text{Square})/G(\text{Hexagon})$ is equal to $3\sqrt{3}/5 = 1.0392$. This is called the quantizing gain of the hexagon over the square. For a given $A(Q)$ (and thus a given $\bar{L}$), the MSE for a hexagonal quantizer is smaller than that for a square quantizer (and thus also a scalar quantizer) by a factor of 1.0392. This is a disappointingly small gain given the added complexity of 2D and hexagonal regions and suggests that uniform scalar quantizers are good choices at high rates.
4.2 Non-uniform 2D quantizers

To be complete, we now analyze the performance of non-uniform 2D quantizers; the analysis is very similar to that of non-uniform scalar quantizers. Consider an arbitrary set of quantization intervals \( \{Q_j\} \). Let \( A(Q_j) \) and \( \text{MSE}_j \) be the area and mean-squared error per dimension respectively of \( Q_j \), i.e.,

\[
A(Q_j) = \int_{Q_j} du ; \quad \text{MSE}_j = \frac{1}{2} \int_{Q_j} \frac{||u - a_j||^2}{A(Q_j)} du
\]

where \( a_j \) is the mean of \( Q_j \). For each region \( R_j \) and \( u \in R_j \), let \( \overline{f}(u) = \text{Pr}(R_j)/A(Q_j) \) be the average pdf in \( R_j \). Then

\[
p_j = \int_{Q_j} f_U(u) du = \overline{f}(u) A(Q_j)
\]

The unconditioned mean-squared error is then

\[
\text{MSE} = \sum_j p_j \text{MSE}_j
\]

Let \( A(u) = A(Q_j) \) and \( \text{MSE}(u) = \text{MSE}_j \) for \( u \in A_j \). Then,

\[
\text{MSE} = \int f_U(u) \text{MSE}(u) du \tag{19}
\]

Similarly,

\[
\mathcal{H}[V] = \sum_j -p_j \log p_j
\]

\[
= \int -f_U(u) \log[\overline{f}(u)A(u)] du
\]

\[
\approx \int -f_U(u) \log[f_U(u)A(u)] du \tag{20}
\]

\[
= 2h[U] - \int f_U(u) \log[A(u)] du \tag{21}
\]

We can again use a Lagrange multiplier to solve for the optimum quantization regions under the high-rate approximation. In particular, from (19) and (21),

\[
\text{MSE} + \lambda \mathcal{H}[V] \approx \lambda 2h[U] + \int_{\mathbb{R}^2} f_U(u) \{\text{MSE}(u) - \lambda \log A(u)\} du \tag{22}
\]

Since each quantization area can be different, we no longer necessarily have a geometric shape whose translates tile the plane. As pointed out earlier, however, the shape that minimizes \( \text{MSE}_c \) for a given quantization area is a circle. Therefore we can lower bound the
MSE, in the Lagrange multiplier, by using this shape. Replacing MSE($u$) by $A(u)/(4\pi)$ in (22), we get

$$\text{MSE} + \lambda H[V] \approx 2\lambda h[U] + \int_{\mathbb{R}^2} f_U(u) \left\{ \frac{A(u)}{4\pi} - \lambda \log A(u) \right\} du$$ (23)

Optimizing for each $u$ separately, we get $A(u) = 4\pi\lambda \log e$. The optimum is achieved where the same size circle is used for each point $u$ (independent of the probability density). This is unrealizable, but still provides a lower bound on the MSE for any given $H[V]$ in the high-rate region. The reduction in MSE over the square region is $\pi/3 = 1.0472$. It appears that the uniform quantizer with hexagonal shape is optimal, but this figure of $\pi/3$ provides a simple bound to the possible gain with 2D quantizers. Either way, the improvement by going to two dimensions is small.

One can now carry out the same sort of analysis for $n$ dimensional quantizers. As $n$ increases, it is possible to get greater gain (i.e., greater reduction in the MSE per dimension for a given entropy per dimension). It is known from a fundamental result in information theory that the maximum gain (i.e., the maximum reduction in MSE) is $\pi e/6 = 1.4233$, which can be approached arbitrarily closely as $n \to \infty$. 